Knotted Surfaces in Dimension Four

Polymath

August 15, 2021

Outline

1. Introduction
2. Group G's Work
3. Group O2's Work
4. Group O3's Work
5. Conclusion

Introduction

Unknots and Unlinks

What is a knotted surface? We must first start with some general knot theory.

Definition

The *unknot* is the simplest knot, defined as an unknotted circle (or something isotopic) which lies in $\mathbb{R}^3$.

Definition

A *c-component unlink* is a diagram which can be untangled into c copies of the unknot.

Tri-plane Diagrams

We can give a two-dimensional representation of a surface K in $\mathbb{R}^4$ via a *tri-plane diagram* [c.f. Figure: 1].

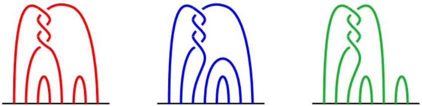


Figure: 1. An example of a tri-plane diagram $K = (K_1, K_2, K_3)$.

A valid tri-plane diagram must satisfy:

- Should have the same number of strands (and hence same number of endpoints on the horizontal axis).
- Each strand should be a trivial tangle, that is, each strand has one relative maximum w.r.t the horizontal axis.
- Each of $K_1 \cup \overline{K_2}$, $K_2 \cup \overline{K_3}$, and $K_3 \cup \overline{K_1}$ should be diagrams for an unlink.

Tri-plane Diagrams (continued)

One can verify that the diagram in Figure: 1 is a tri-plane diagram. In fact, it depicts the knotted surface known as the spun trefoil.

Clearly, each tangle is trivial and each diagram has four strands. Inspecting the below figure allows us to verify that each mirrored union is an unlink.

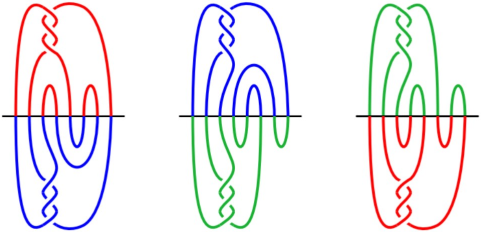


Figure: 2. A verification that the mirrored unions are unlinks.

Tri-plane Moves

Just as in 3-dimensional knot theory, tri-plane diagrams can undergo local changes via the three Reidemeister moves, which are pictured in Figure 3.

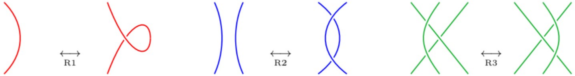


Figure: 3. The three Reidemeister moves.

Additionally, we can perform mutual braid moves, which are shown in Figure 4.

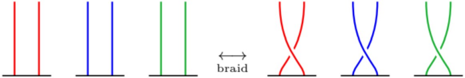


Figure: 4. A mutual Braid Move

Tri-plane moves (continued)

We can also perform stabilization and destabilization moves as pictured in Figure 5.

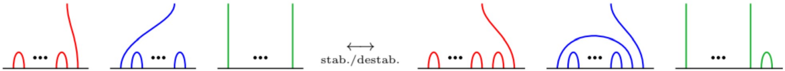


Figure: 5. Stabilization and Destabilization

Definition

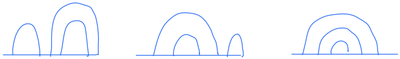
A *tri-plane move* is either a Reidemeister move, a mutual braid move, or a stabilization/destabilization.

Tri-plane Diagrams for Unknotted Surfaces

- Diagram for P^+

- Diagram for P^-

- Diagram for T



- Diagram for U

Unknotted Surfaces

- We write $K \sim K'$ if K' can be obtained from K via a finite series of tri-plane moves and planar isotopies. One can check that $\sim$ as defined above is an equivalence relation with equivalence class $[K]$. We call $[K]$ a *knotted surface.*
- We call a surface *unknotted* if it is either the unknotted two-sphere U, or a finite connected sum of copies of the positive unknotted projective plane P^+, the negative unknotted projective plane P^-, or the unknotted torus, T.

Writhe

Definition

Take a knot (or link) diagram and assign an orientation by moving along the knot with arrows. To each crossing where the arrow in the right direction is an over crossing, associate a $+1$. To each crossing where the arrow pointing right is an under crossing, associate a -1. We define the *writhe* of K, $w(K)$ as the sum of these $+1$ and -1 which appear at each crossing.

Figure 6.8 Determine the writhe of this link projection.

Bridge Number and Crossing Number

Definition

We define the *bridge number* $b(K)$ of a tri-plane diagram K as the number of strands in each picture. The bridge number of an unknotted surface $b([K])$ is defined to be the minimal bridge number that appears for the given surface, i.e. $b([K]) = \min\{b(K') : K \sim K'\}$.

Definition

We define the *crossing number* $c(K)$ of a tri-plane diagram K is the number of times that strands in all three diagrams cross. The crossing number of a knotted surface $c([K])$ is defined as the minimal crossing number that appears in some representation of the surface K, i.e.e $c([K]) = \min\{c(K') : K' \sim K\}$.

Patch Numbers and Invariants

Definition

We define the *patch numbers* p_{12}, p_{23}, and p_{31} as the number of components which appear in the unlinks for $K_1 \cup \overline{K_2}$, $K_2 \cup \overline{K_3}$, and $K_3 \cup \overline{K_1}$, respectively.

Definition

We define the *Euler characteristic* of a surface K as $\chi(K) = p_{12} + p_{23} + p_{31} - b$, where p_{ij} are the patch numbers and b is the bridge number. We call χ an invariant because if $K \sim K'$, then $\chi(K) = \chi(K')$.

Definition

We define the *normal Euler number* of a surface K as $e(K) = w(K_1 \cup \overline{K_2}) + w(K_2 \cup \overline{K_3}) + w(K_3 \cup \overline{K_1})$. We call e and invariant because if $K \sim K'$, then $e(K) = e(K')$.

Orientability

Definition

Given a tri-plane diagram K with b strands, to each endpoint we will associate either a $+$ or a $-$. We do so in such a way so that each strand has a $+$ at one endpoint and a $-$ at the other. We also ensure that the signs are consistent between all three diagrams. If all $2b$ signs in each diagram match and each strand receives one sign each, then we say that K is *orientable*, and we write $o(K) = 1$. Otherwise, $o(K) = 0$. It is worth noting that o is also an invariant.

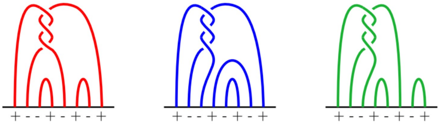


Figure: 7. The spun trefoil is orientable.

Connected Sums

Another definition:

Definition

Consider two tangle diagrams K_1 and K_2. We can take a connected sum $K_1 \# K_2$ of the diagrams by detaching an endpoint on each diagram from the y-axis and connecting them. See the below figure.

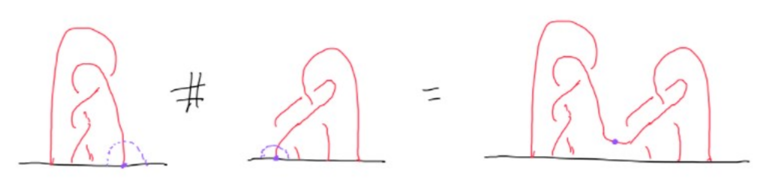
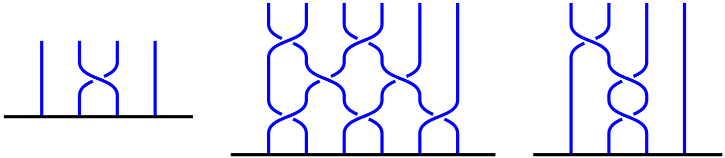


Figure: 8. Connected sum of two tangle diagrams.

Group G's Work

Representing Tangles

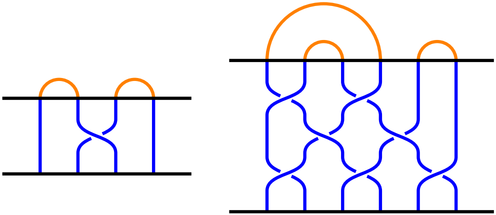
- Need to tell a computer what a tangle is
- Snappy/Spherogram are good, but not what we need
- Idea: a trivial tangle is a *braid* plus a *cap diagram*
- Braids: form a group; Sage understands this group



Representing Tangles

- Cap diagram:

- Cap diagrams are generated using a classical algorithm
- Capping off braids:



Listing Triplane Diagrams

Goal

List all triplane diagrams with small bridge & crossing #.

- How do we check if three trivial tangles form a triplane diagram?
  1. Translate to Spherogram
  2. Check if resulting links are unlinks
- Problem: this takes a *really* long time
  - There are A LOT of braids
  - Looking at three tangles cubes everything
- Solutions:
  - Braids are words in generators, so remove duplicates
  - Generation and verification separately

Computing Invariants

Goal

Automatically compute invariants of a triplane diagram

- Orientability and Euler characteristic: variations on greedy algorithm
  - Start at an arbitrary point/orientation
  - Chase around the diagrams making inferences
- Other invariants: Snappy/Spherogram
  - E.g., computing writhe

Group O2's Work

Some Preliminary Results

- **Proposition 1** $[T\#P^+] = [P^+\#P^-\#P^+]$,$[T\#P^-] = [P^-\#P^+\#P^-]$
- **Remark** If an unknotted P is orientable, then $[P] = [U]$ or $[P] = [T^n]$; otherwise $[P] = [P^+]^n\#[P^-]^m =: P^{n,m}$.
- **Proposition 2** For any tri-plane diagram P, $|e(P)| \leq 2c(P)$.
- **Corollary 3** $|n - m| \leq c(P^{n,m}) \leq max\{n, m\}$.
- **Proposition 4** $c([P\#P']) \leq c([P]) + c([P'])$.
- **Proposition 5** If $c([P]) = 0$, then $[P]$ is orientable.

Main Results

Theorem

$c([P^{n,m}]) = max\{1, |n - m|\}$

Proof.

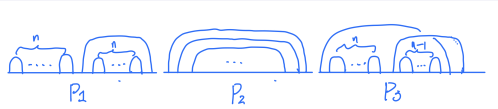
- **Step I.** Prove $c([P^{n,m}]) = c([P^{m,n}])$
- **Step II.** Prove $c([P^{n,n}]) = c([P^{n,n+1}]) = 1$
- **Step III.** Apply Prop. 4:
  $c([P^{n,n+k}]) \leq c([P^{n,n+1}]) + (k-1)c([P^{0,1}]) = 1 + (k-1) = k$

$\square$

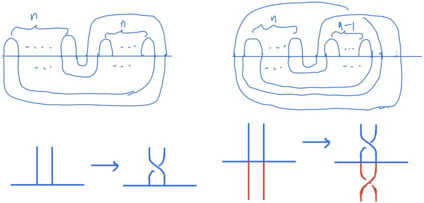
Step II

Lemma

$\forall n \in \mathbb{N}_+, [P^{n,n}]$ has a diagram that looks like the following:



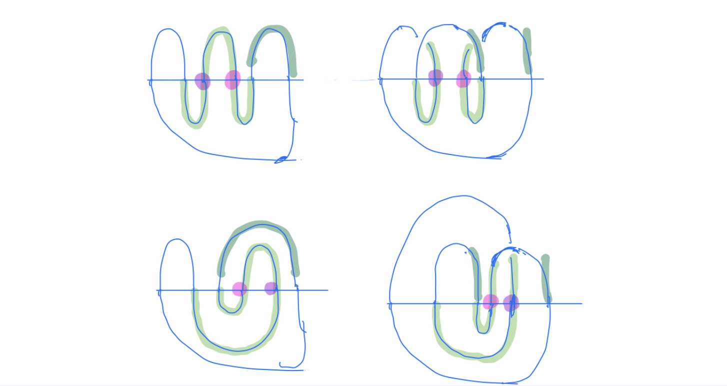
- We use the pair of unlink diagrams, $P_{12} = P_1 \cup \overline{P_2}$ and $P_{32} = P_3 \cup \overline{P_2}$ to visualize the tri-plane moves to reduce crossings.



Step II

Proof.

- Base case $n = 1$:

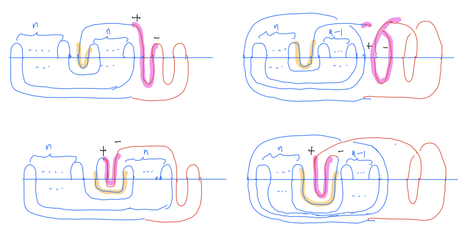


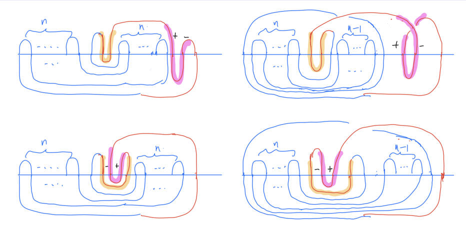
□

Step II

Proof.

- Inductive Step from n to $n+1$:





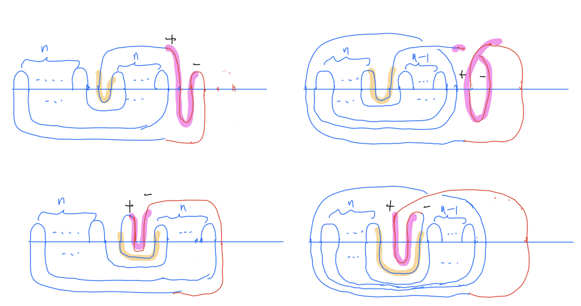
Step II

Corollary

$c([P^{n,n+1}]) = 1$

Proof.

- Perform a left translation as follows



□

Future Work

Conjecture

If $c([P]) < 4$, then $[P]$ is unknotted.

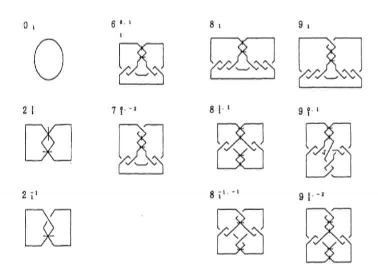
Group O3's Work

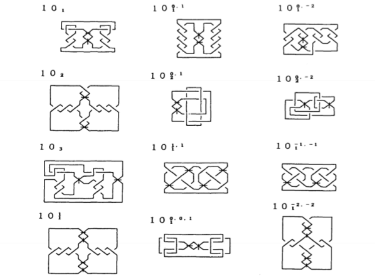
Goal: Compile a Body of Knotted Surfaces

- Examples are immensely helpful in the research process as they build intuition and are critical for testing conjectures.
- In regular knot theory, it is easy to work with and come up with new knots.
- Unfortunately, coming up with a list of distinct and interesting tri-plane diagrams for knotted surfaces is not easy.

CH-Diagrams

Luckily, there are already lists of knotted surfaces.

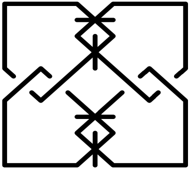


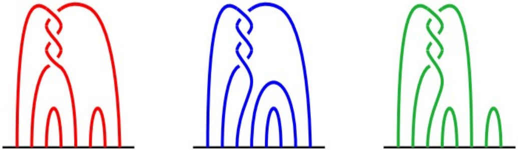


Converting CH-Diagrams

Here we see a ch-diagram and a tri-plane diagram for the spun trefoil knotted surface.

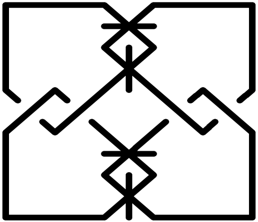
8^{1}-Spun Trefoil

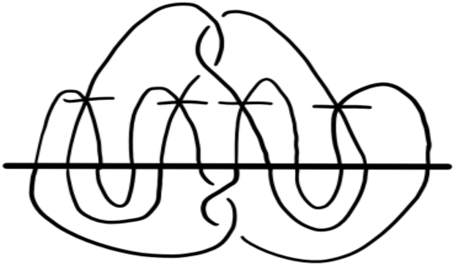




Converting CH-Diagrams

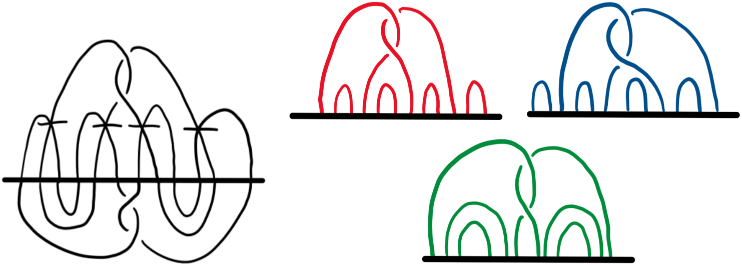
To convert a CH-diagram, we first want to put it into bridge position.





Converting CH-Diagrams

We then resolve the top portion into two tangles, and the bottom into it's own.



Converting CH-Diagrams

- We then verify if the tangles make up a valid tri-plane diagram.
- If they don't, we have to start over and try again with a different bridge position.
- Once we have a tri-plane diagram we can calculate invariants for the diagram.

Conclusion

Acknowledgements

Big thanks to...

- Alex Zupan, Ana Wright, and Nick Meyer for guidance and insight
- Polymath REU organizers for this wonderful opportunity!

References

J. Meier & A. Zupan. Bridge trisections of knotted surfaces in S^4. 2017. *Transactions of the American Mathematical Society* **369**, 7343-7386. https://doi.org/10.1090/tran/6934

H. Yoshikawa. An enumeration of surfaces in four-space. 1994. Osaka Journal of Mathematics **31**(3): 497-522.