Completeness of Positive Linear Recurrence Sequences

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Introduction

Positive Linear Recurrence Sequences

Definition

A sequence $\{H_i\}_{i\geq 1}$ of positive integers is a **Positive Linear Recurrence Sequence (PLRS)** if the following properties hold:

- (Recurrence relation) There are non-negative integers $L, c_1, \ldots, c_L$ such that

$$H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$$

  with L, c_1, c_L positive.
- (Initial conditions) $H_1 = 1$, and for $1 \leq n < L$,

$$H_{n+1} = c_1 H_n + \cdots + c_n H_1 + 1$$

Positive Linear Recurrence Sequences

- We write $[c_1, \dots, c_L]$ for $H_{n+1} = c_1 H_n + \cdots + c_L H_{n-L+1}$.
- For example, for the Fibonacci numbers, we write $[1, 1]$. This definition gives initial conditions $F_1 = 1, \ F_2 = 2$.
- Despite satisfying positive linear recurrences, the Lucas and Pell numbers are not PLRS, since their initial conditions do not meet the definition.

Introduction to Completeness

Definition

A sequence $\{H_i\}_{i\geq 1}$ is called **complete** if every positive integer is a sum of its terms, using each term at most once.

- The sequence with the recurrence $[1, 3]$ is *not* complete. Its terms are $\{1, 2, 5, 11, \dots\}$; you cannot get 4 or 9 as the sequence grows too quickly.
- The Fibonacci sequence $F_{n+1} = F_n + F_{n-1}$, with initial conditions $F_1 = 1$, $F_2 = 2$, is complete (follows from Zeckendorf's Theorem).

The Doubling Sequence

The PLRS [2], which has the recurrence $H_{n+1} = 2H_n$, has terms $H_n = 2^{n-1}$ and is complete because every integer has a binary representation.

Theorem (Brown)

The complete sequence with maximal terms is $H_n = 2^{n-1}$.

Any PLRS of the form $[1, \ldots, 1, 2]$ has the same terms as [2], i.e., $H_n = 2^{n-1}$.

Brown's Criterion

Theorem (Brown)

A nondecreasing sequence $\{H_i\}_{i\geq 1}$ is complete if and only if $H_1 = 1$ and for every $n \geq 1$,

$$H_{n+1} \leq 1 + \sum_{i=1}^{n} H_i.$$

Can we bound where a sequence must fail Brown's Criterion? We think so!

Conjecture (SMALL 2020)

If a PLRS $H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$ incomplete, then it fails Brown's criterion before the $2L$-th term.

Families of Sequences

Analyzing Families of Sequences

Theorem (SMALL 2020)

1. $[1, \underbrace{0, \dots, 0}_{k}, N]$, *is complete if and only if*

$$N \leq \left\lfloor \frac{(k+2)(k+3)}{4} + \frac{1}{2} \right\rfloor .$$

2. $[1, 1, \underbrace{0, \dots, 0}_{k}, N]$, *is complete if and only if*

$$N \leq \left\lfloor \frac{F_{k+6} - (k+5)}{4} \right\rfloor ,$$

where F_k is the kth Fibonacci number.

Proof Sketch

Theorem (SMALL 2020)

1. *$[1, 0, \ldots, 0, N]$, with k zeros, is complete if and only if $N \leq \left\lfloor \frac{(k+2)(k+3)}{4} + \frac{1}{2} \right\rfloor$.*

Partial Proof. We sketch that if $N_{\max} = \left\lfloor \frac{(k+2)(k+3)}{4} + \frac{1}{2} \right\rfloor$, then the sequence is complete. It is similar for $N < N_{\max}$. With the recurrence relation and Brown's Criterion,

$$\begin{aligned} H_{n+1} &= H_n + N_{\max} H_{n-k-1} \\ &\leq H_n + (N_{\max} - 1) H_{n-k-1} + H_{n-k-2} + \cdots + H_1 + 1 \end{aligned}$$

By induction, $(N_{\max} - 1) H_{n-k-1} \leq H_{n-1} + \cdots + H_{n-k-1}$, so

$$\leq H_n + \cdots + H_1 + 1.$$

Example for $L = 6$

By the previous theorem, $[1, 0, 0, 0, 0, N]$ is complete for $N \leq 11$.

Question

Does there exist a complete PLRS of length $L = 6$ with $N > 11$?

Example for $L = 6$

Here are the maximal last terms for preserving completeness for several other sequences of length $L = 6$:

- $[1, 0, 0, 0, 0, N]$ is complete for $N \leq 11$.
- $[1, 1, 0, 0, 0, N]$ is complete for $N \leq 11$.
- $[1, 0, 1, 0, 0, N]$ is complete for $N \leq 12$.
- $[1, 0, 0, 1, 0, N]$ is complete for $N \leq 11$.
- $[1, 0, 0, 0, 1, N]$ is complete for $N \leq 10$.

Why is $[1, 0, 1, 0, 0, 12]$ complete, but $[1, 0, 0, 0, 0, 12]$ is not complete?

Example for $L = 6$

Why is $[1, 0, 1, 0, 0, 12]$ complete, but $[1, 0, 0, 0, 0, 12]$ is not complete?

- $[1, 0, 0, 0, 0, 12]$ has terms $\{1, 2, 3, 4, 5, 6, 18, 42, \dots\}$ and so computing the sums $\sum_{i=1}^{n} H_i + 1$ we see $\{2, 4, 7, 11, 16, 22, 40, \dots\}$
- $[1, 0, 1, 0, 0, 12]$ has terms $\{1, 2, 3, 5, 8, 12, 29, 61, \dots\}$ and so computing the sums $\sum_{i=1}^{n} H_i + 1$ we see $\{2, 4, 7, 12, 20, 32, 61, \dots\}$
- $[1, 1, 1, 0, 0, 12]$ has terms $\{1, 2, 4, 8, 15, 28, 63, \dots\}$ and so computing the sums $\sum_{i=1}^{n} H_i + 1$ we see $\{2, 4, 8, 16, 31, 59, \dots\}$

Sequences of Initial Ones

Theorem (SMALL 2020)

If a sequence $[\underbrace{1,\ldots,1}_{m},\underbrace{0,\ldots,0}_{k},N]$ *is complete with* $m \geq 3$, *then*

$$N \leq \frac{1}{2}\left(1+\sum_{i=1}^{k+1} F_i^{(m)} + \sum_{i=1}^{k+1-m} F_i^{(m)} + \cdots + \sum_{i=1}^{(k+1) \bmod m} F_i^{(m)}\right)$$

where $F_i^{(m)}$ *is the m-bonacci sequence,* $[\underbrace{1,\ldots,1}_{m}]$.

Theorem on Adding Ones

Theorem (SMALL 2020)

- *For $L \geq 6$, consider the sequence $\{H_n\}$ given by $[1, 0, \dots, 0, 1, 0, \dots, 0, M]$. Then, if M is maximal such that $\{H_n\}$ is complete, and N is maximal such that $[1, 0, \dots, 0, N]$ is complete, we have $M \geq N$.*
- *For a fixed length L, the sequence $[1, \underbrace{0, \dots, 0}_{k}, \underbrace{1, \dots, 1}_{m}, N]$ with m ones has a lower bound on N than the sequence $[1, \underbrace{0, \dots, 0}_{k-1}, \underbrace{1, \dots, 1}_{m+1}, N]$.*

  In particular, if $m < \frac{L}{2}$, the bound is precisely

$$N \leq \left\lfloor \frac{(L-m)\,(L+m+1)}{4} + \frac{1}{48}m(m+1)(m+2)(m+3) + \frac{1-2m}{2} \right\rfloor .$$

Modifying Sequences

Modifying Coefficients of a PLRS

When studying a PLRS, what modifications to the recurrence coefficients preserve completeness or incompleteness?

Theorem (SMALL 2020)

- *If a sequence* $[c_1, \ldots, c_{L-1}, c_L]$ *is complete, then so is* $[c_1, \ldots, c_{L-1}, d_L]$ *for any* $d_L \leq c_L$.
  Remark. This is not true for c_i *in any position.*
- *If a sequence* $[\underbrace{1, \ldots, 1}_{m}, \underbrace{0, \ldots, 0}_{k}, c_L]$ *is complete and* $c_L = 2^{k+1} - 1$, $[\underbrace{1, \ldots, 1}_{m}, \underbrace{0, \ldots, 0}_{k}, c_L + j]$ *is incomplete for any positive integer* j.

Modifying Lengths of a PLRS

Theorem (SMALL 2020)

- *If a sequence* $[c_1, \dots, c_L]$ *is incomplete, then so is* $[c_1, \dots, c_{L-1} + c_L]$.
- *If a sequence* $[c_1, \dots, c_L]$ *is incomplete, then so is* $[c_1, \dots, c_L, c_{L+1}]$ *for any* $c_{L+1} > 0$.

Conjecture (SMALL 2020)

If a sequence $[\underbrace{1, \dots, 1}_{m}, \underbrace{0, \dots, 0}_{k}, c_L]$ *is complete, then so is*

$[\underbrace{1, \dots, 1}_{m+j}, \underbrace{0, \dots, 0}_{k}, c_L]$ *for any positive integer* j.

Principal Roots

Principal Roots

Theorem (Binet's Formula)

If $r_1, \ldots, r_k$ are the distinct roots of the characteristic polynomial of a PLRS $\{H_n\}$, then there exist polynomials $q_1, \ldots, q_k$ such that $H_n = q_1(n)r_1^n + \cdots + q_k(n)r_k^n$.

For PLRS, the characteristic polynomial has a unique positive root r_1 which is the largest in absolute value, called the *principal root.*

Theorem (SMALL 2020)

If H_n is a complete PLRS and r_1 is its principal root, then $r_1 \leq 2$.

Bounding Principal Roots

- If a sequence is complete, $r_1 \leq 2$.
- There exists a second bound $1 < B_L < 2$ on the principal roots, so that if a sequence is incomplete, the its principal root r_1 satisfies $r_1 \geq B_L$.. This bound is dependent on the length of the generating sequence $[c_1, \dots, c_L]$. We conjecture the following:

Conjecture (SMALL 2020)

For any given L, the incomplete sequence of length L with the lowest principal root is $[1, 0, \dots, 0, \left\lceil \frac{L(L+1)}{4} \right\rceil + 1]$.

- If this holds, then for large L, we would have $B_L \approx (L/2)^{2/L}$. In particular, $\lim_{L\to\infty} B_L = 1$.

Root-Bounding Proof Sketch

Conjecture (SMALL 2020)

For any given L, the incomplete sequence of length L with the lowest principal root is $[1, 0, \ldots, 0, \left\lceil \frac{L(L+1)}{4} \right\rceil + 1]$.

Suppose $[c_1, \ldots, c_L]$ is an incomplete sequence.

Case 1: $\sum_{k=1}^{L} c_k \geq 2 + \left\lceil \frac{L(L+1)}{4} \right\rceil$

We combine the following two invariant arguments:

- The principal root of $[c_1, \ldots, c_L]$ is strictly greater than that of $[c_1, \ldots, c_k - 1, \ldots, c_L + 1]$, for any k.
- The principal root of $[1, 0, \ldots, 0, S]$ is strictly greater than that of $[1, 0, \ldots, 0, S - 1]$.

Combining these two, any sequence with large sum can be "reduced" to $[1, 0, \ldots, 0, \left\lceil \frac{L(L+1)}{4} \right\rceil + 1]$.

Root-Bounding Proof Sketch

Case 2: $\sum_{k=1}^{L} c_k \leq 1 + \left\lceil \frac{L(L+1)}{4} \right\rceil$

It can be shown any "counterexample" would fulfill:

- $\forall 1 \leq k \leq L+1$,

$$\sum_{i=2}^{k} c_i \leq \left\lceil \frac{k(k+1)}{4} \right\rceil .$$

- $\sum_{i=2}^{L} c_i \left(\lambda_{L+1}^{L+1-i} - \lambda_L^{L-i}\right) < \frac{L+2}{2}$, where λ_L is the root of $[1, 0, \ldots, 0, \lceil L(L+1)/4 \rceil + 1$.

This forces the coefficients of $[c_1, \ldots, c_L]$ to be small enough to force a contradiction; for example, an analytical argument shows the first 32.5% or so must be 0.

Future Directions

Future Directions

- Extend analysis of the bound of N in $[\underbrace{1,\ldots,1}_{m},0,\ldots,0,N]$, which involves the m-bonacci numbers, defined by $[\underbrace{1,\ldots,1}_{m}]$.
- Find the bound N for arbitrary coefficients $c_2,\ldots,c_{L-1}$ in $[1,c_2,\ldots,c_{L-1},N]$.
- Prove the conjectures made in this presentation.

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