Introduction to Completeness of Generalized Fibonacci Sequences

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Introduction

Motivation

- **Positive linear recurrence sequences** (PLRS) generalize the Fibonacci numbers in Zeckendorf's theorem.

- **Complete** sequences can be used to to express integers using sums of terms.

Research Question

How can we determine whether a PLRS is complete based on the coefficients in its defining recurrence relation?

Positive Linear Recurrence Sequences

Definition

A sequence $\{H_i\}_{i\geq 1}$ of positive integers is a **Positive Linear Recurrence Sequence (PLRS)** if:

- (Recurrence relation) There are non-negative integers $L, c_1, \ldots, c_L$ such that

$$H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$$

  with L, c_1, c_L positive.
- (Initial conditions) $H_1 = 1$, and for $1 \leq n \leq L$,

$$H_{n+1} = c_1 H_n + \cdots + c_n H_1 + 1$$

Positive Linear Recurrence Sequences

- Write $[c_1, \dots, c_L]$ for $H_{n+1} = c_1 H_n + \cdots + c_L H_{n-L+1}$.
- Fibonacci numbers: $[1, 1]$. Initial conditions $F_1 = 1,\ F_2 = 2$.
- (Lucas and Pell numbers are not PLRS, due to initial conditions).

Definition

A sequence $\{H_i\}_{i\geq 1}$ is **complete** if every positive integer is a sum of its terms, using each term at most once.

- The sequence $[1, 3]$ is *not* complete. Its terms are $\{1, 2, 5, 11, \dots\}$; you cannot get 4 or 9.
- The Fibonacci sequence is complete (follows from Zeckendorf's Theorem).

The Doubling Sequence $H_{n+1} = 2H_n$

The PLRS [2] has terms $H_n = 2^{n-1}$, i.e., $\{1, 2, 4, 8, \ldots\}$, and is complete (every integer has a binary representation).

Theorem (Brown)

The complete sequence with maximal terms is $H_n = 2^{n-1}$.

Any PLRS of the form $[1, \ldots, 1, 2]$ has the same terms as [2], i.e., $H_n = 2^{n-1}$.

Brown's Criterion

Theorem (Brown)

A nondecreasing sequence $\{H_i\}_{i\geq 1}$ *is complete if and only if* $H_1 = 1$ *and for every* $n \geq 1$,

$$H_{n+1} \;\leq\; 1 + \sum_{i=1}^{n} H_i.$$

Definition

The ***n*-th Brown's Gap** of a sequence $\{H_i\}_{i\geq 1}$ is

$$B_{H,n} \;:=\; 1 + \left(\sum_{i=1}^{n-1} H_i\right) - H_n.$$

Modifying Sequences

Example for $L = 6$

Example

$[1, 0, 0, 0, 0, N]$ is complete if and only if $N \leq 11$.

Question

Is there another choice of coefficients $[c_1, \ldots, c_5, N]$, that generates a complete PLRS, with some $N > 11$?

Example for $L = 6$

- $[1, 0, 0, 0, 0, N]$ is complete for $N \leq 11$.
- $[1, 1, 0, 0, 0, N]$ is complete for $N \leq 11$.
- $[1, 0, 1, 0, 0, N]$ is complete for $N \leq 12$.
- $[1, 0, 0, 1, 0, N]$ is complete for $N \leq 11$.
- $[1, 0, 0, 0, 1, N]$ is complete for $N \leq 10$.

Why is $[1, 0, 1, 0, 0, 12]$ complete, but $[1, 0, 0, 0, 0, 12]$ is not complete?

Example for $L = 6$

Why is $[1, 0, 1, 0, 0, 12]$ complete, but $[1, 0, 0, 0, 0, 12]$ is not complete?

- $[1, 0, 0, 0, 0, 12]$ has terms $\{1, 2, 3, 4, 5, 6, 18, 42, \dots\}$ and so computing $1 + \sum_{i=1}^{n} H_i$ we see $\{2, 4, 7, 11, 16, 22, 40, \dots\}$
- $[1, 0, 1, 0, 0, 12]$ has terms $\{1, 2, 3, 5, 8, 12, 29, 61, \dots\}$ and so computing $1 + \sum_{i=1}^{n} H_i$ we see $\{2, 4, 7, 12, 20, 32, 61, \dots\}$
- $[1, 1, 1, 0, 0, 12]$ has terms $\{1, 2, 4, 8, 15, 28, 63, \dots\}$ and so computing $1 + \sum_{i=1}^{n} H_i$ we see $\{2, 4, 8, 16, 31, 59, \dots\}$

Modifying Coefficients of a PLRS

What modifications to the coefficients preserve completeness or incompleteness?

Theorem (SMALL 2020)

If $[c_1, \ldots, c_L]$ is any incomplete sequence, then the sequence $[c_1, \ldots, c_{L-2}, c_{L-1} + c_L]$ is also incomplete.

Theorem (SMALL 2020)

If a sequence $[c_1, \ldots, c_{L-1}, c_L]$ is complete, then so is $[c_1, \ldots, c_{L-1}, d_L]$ for any $1 \leq d_L \leq c_L$.
Remark. Not true for c_i in an arbitrary position.

We discuss bounds for the last coefficient.

Families of Sequences

Analyzing Families of Sequences

Theorem (SMALL 2020)

- $[1, \underbrace{0, \dots, 0}_{k}, N]$, *is complete if and only if*

$$N \leq \left\lfloor \frac{(k+2)(k+3)}{4} + \frac{1}{2} \right\rfloor.$$

- $[1, 1, \underbrace{0, \dots, 0}_{k}, N]$, *is complete if and only if*

$$N \leq \left\lfloor \frac{F_{k+6} - (k+5)}{4} \right\rfloor,$$

where F_k is the kth Fibonacci number.

Proof Sketch

Theorem

$[1, 0, \ldots, 0, N]$, *with* k *zeros, is complete if and only if* $N \leq \left\lfloor \frac{(k+2)(k+3)}{4} + \frac{1}{2} \right\rfloor$.

Partial Proof. We sketch that if $N_{\max} = \left\lfloor \frac{(k+2)(k+3)}{4} + \frac{1}{2} \right\rfloor$, then the sequence is complete.

With the recurrence relation and Brown's criterion,

$$\begin{aligned} H_{n+1} &= H_n + N_{\max} H_{n-k-1} \\ &\leq H_n + (N_{\max} - 1) H_{n-k-1} + H_{n-k-2} + \cdots + H_1 + 1 \end{aligned}$$

By induction, $(N_{\max} - 1) H_{n-k-1} \leq H_{n-1} + \cdots + H_{n-k-1}$, so

$$\leq H_n + \cdots + H_1 + 1.$$

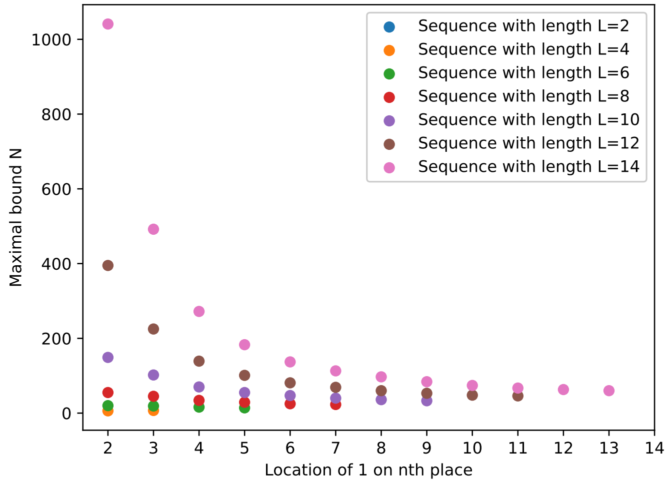

Figure 1: $[1, \underbrace{0, \ldots, 0}_{k}, 1, \underbrace{0, \ldots, 0}_{g}, N]$ with location of middle one varying, where each color represents a fixed length L.

Theorem on Switching Ones

Theorem (SMALL 2020)

Let $L \geq 6$ *fixed and* $\{H_n\} = [1, \underbrace{0, \dots, 0}_{L-g-3}, 1, \underbrace{0, \dots, 0}_{g}, M]$, $0 < g \leq L - 3$. *If* M *is maximal such that* $\{H_n\}$ *is complete, and* N *is maximal such that* $[1, 0, \dots, 0, N]$ *is complete,* $M \geq N$.

In particular,

- $[1, 0, \dots, 0, 0, 1, M]$ *is complete if and only if* $M \leq N - 1$
- $[1, 0, \dots, 0, 1, 0, M]$ *is complete if and only if* $M \leq N$.

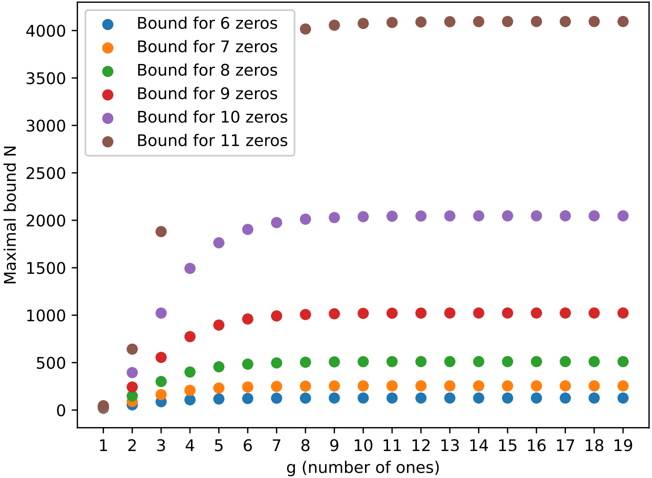

Figure 2: $[\underbrace{1,\ldots,1}_{g},\underbrace{0,\ldots,0}_{k},N]$ with k and g varying, where each color represents a fixed k.

Sequences of Initial Ones

Theorem (SMALL 2020)

If a sequence $[\underbrace{1,\ldots,1}_{g},\underbrace{0,\ldots,0}_{k},N]$ *is complete with* $g \geq 3$,

then

$$N \leq \frac{1}{2}\left(1+\sum_{i=1}^{k+1} F_i^{(g)}+\sum_{i=1}^{k+1-g} F_i^{(g)}+\cdots+\sum_{i=1}^{(k+1) \bmod g} F_i^{(g)}\right)$$

where $F_i^{(g)}$ *is the* g*-bonacci sequence,* $[\underbrace{1,\ldots,1}_{g}]$.

Sequences of Initial Ones

Conjecture (SMALL 2020)

If a sequence $[\underbrace{1, \ldots, 1}_{g}, \underbrace{0, \ldots, 0}_{k}, N]$ is complete, then so is $[\underbrace{1, \ldots, 1}_{g+j}, \underbrace{0, \ldots, 0}_{k}, N]$ for any positive integer j.

Theorem (SMALL 2020)

Consider $[\underbrace{1, \ldots, 1}_{g}, \underbrace{0, \ldots, 0}_{k}, N]$.

- *For* $g \geq k + \lceil \log_2 k \rceil$, *the bound on* N *is* $N \leq 2^{k+1} - 1$
- *For* $k \leq g < k + \lceil \log_2 k \rceil$, *the bound on* N *is* $N \leq 2^{k+1} - \left\lceil \frac{k}{2^{g-k}} \right\rceil$

The $2L - 1$ conjecture

The $2L-1$ conjecture

Can we bound where a sequence must fail Brown's Criterion?
We think so!

Conjecture (SMALL 2020)

If a PLRS $H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$ incomplete, then it fails Brown's criterion before the $2L$-th term.

The closest we've gotten:

Theorem (SMALL 2020)

The PLRS $\{H_i\}_{i\geq 1}$ generated by $[c_1, \ldots, c_L]$ is complete if

$$\begin{cases} B_{H,n} \geq 0, \ 1 \leq n < L \\ B_{H,n} > 0, \ L \leq n \leq 2L-1 \end{cases}$$

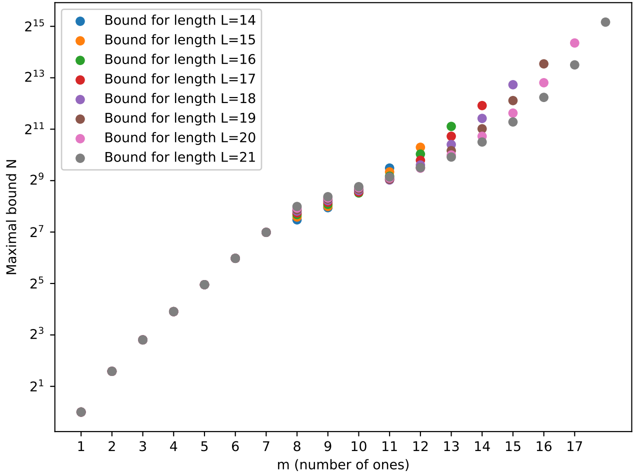

Figure 3: $[1, \underbrace{0, \ldots, 0}_{k}, \underbrace{1, \ldots, 1}_{m}, N]$ with number of ones (m) varying, depending on L.

Conditional result on Adding Ones

If the $2L - 1$ conjecture holds, we have the following:

Theorem (SMALL 2020)

For a fixed length L, the sequence $[1, \underbrace{0, \ldots, 0}_{k}, \underbrace{1, \ldots, 1}_{m}, N]$ with m ones has a lower bound on N than the sequence $[1, \underbrace{0, \ldots, 0}_{k-1}, \underbrace{1, \ldots, 1}_{m+1}, N]$.

In particular, if $m < \frac{L}{2}$, the bound is precisely

$$N \leq \left\lfloor \frac{(L-m)(L+m+1)}{4} + \frac{1}{48}m(m+1)(m+2)(m+3) + \frac{1-2m}{2} \right\rfloor .$$

Binet's Formula and Generalizations

Characteristic Polynomials

Definition

For a PLRS $\{H_n\}$ defined by $[c_1, \dots, c_L]$, define the characteristic polynomial

$$p(x) = x^L - \sum_{i=1}^{L} c_i x^{L-i}.$$

- By Descartes's Rule of Signs, $p(x)$ there is one positive real root, the **principal root**.
- The principal root is always the largest: for any root $z \in \mathbb{C}$, $|z| < r$.

Generalized Binet's Formula

Theorem (Generalized Binet's Formula)

If $r_1, \dots, r_k$ are the roots of the polynomial of a linear recurrence $\{H_n\}$ with multiplicities $m_1, \dots, m_k$, there are polynomials $q_1, \dots, q_k$ with $\deg(q_i) \leq m_i - 1$ such that

$$H_n = q_1(n)r_1^n + \ldots + q_k(n)r_k^n.$$

- If $\{H_n\}$ is a PLRS, let r_1 be the principal root; since $m_1 = 1$ and for all $i, r_1 > |r_i|$, then $H_n = \Theta(r_1^n)$.
- Complete sequences should grow "slowly". Can we relate the size of r_1 to completeness?

Bounding the Principal Root

First Bounds on r_1

Recall $p(x) = x^L - \sum_{i=1}^{L} c_i x^{L-i}$.

As $c_L \geq 1$, we know $r_1 > 1$. ($c_L = \prod r_i^{m_i}$, and r_1 is the biggest root by magnitude).

Lemma (SMALL 2020)

If H_n is a complete PLRS and r_1 is its principal root, then $r_1 \leq 2$.

Proof.

Otherwise, as $H_n = \Theta\left(r_1^n\right)$, for large n our terms would exceed the maximal sequence $\{2^{n-1}\}$. □

Note: there are incomplete sequences with principal roots $r \leq 2$.

Is 2 a Useful Bound?

- We can find complete sequences with roots of sizes arbitrarily close to 2. (Sequences of the form $[\underbrace{1, \ldots, 1}_{L}]$.)
- Checking $r_1 \leq 2$ is a fast method to eliminate candidates for completeness.
- $p(x) = x^L - \sum_{i=1}^{L} c_i x^{L-i}$ has one positive real root, and $p(x) > 0$ for large x, so $r_1 \leq 2$ if and only if $p(2) \geq 0$. This is much faster than checking Brown's Criterion!

Lower Bound

Lemma (SMALL 2020)

For any L, there exists a second bound B_L such that if a sequence $[c_1, \ldots, c_L]$ is incomplete, then $r_1 \geq B_L$.

Proof.

- There are finitely many sequences $[c_1, \ldots, c_L]$ with $p(2) = 2^L - \sum_{i=1}^{L} c_i 2^{L-i} \geq 0$.
- Hence finitely many incomplete sequences with $r_1 \leq 2$, so just find the minimum root - B_L.

$\square$

We now aim to determine the precise values of B_L.

The Minimal Incomplete Sequence

Theorem (SMALL 2020)

$[1, \underbrace{0, \ldots, 0}_{L-2}, N]$, *is complete if and only if*

$$N \leq \left\lceil \frac{L(L+1)}{4} \right\rceil .$$

Conjecture (SMALL 2020)

For any L, the incomplete sequence of length L with smallest principal root is $[1, 0, \ldots, 0, \left\lceil \frac{L(L+1)}{4} \right\rceil + 1]$.

- Let λ_L the principal root of $[1, 0, \ldots, 0, \left\lceil \frac{L(L+1)}{4} \right\rceil + 1]$. This is saying $\lambda_L = B_L$, for all L.

Denseness of Incomplete Roots

Theorem (SMALL 2020)

For any $L \in \mathbb{Z}^+$, let R_L be the set of roots of all incomplete PLRS of length L. Then, for any $\varepsilon > 0$, there exists an M such that for all $L > M$, for any ε-ball $B_\varepsilon \subset [1, 2]$, $B_\varepsilon \cap R_L \neq \varnothing$.

Corollary

The set $R = \bigcup_{L=1}^{\infty} R_L$ of all principal roots of incomplete sequences is dense in $[1, 2]$.

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- Thank you. Any questions?

Appendix

Proof of Denseness Theorem

We use that the λ_L roots are decreasing, and $\lim_{L\to\infty} \lambda_L = 1$.

Proof.

Consider the following incomplete sequences:

$$[1, 0, \dots, 0, \left\lceil \frac{L(L+1)}{2} \right\rceil + 1],\ [1, 0, \dots, 0, \left\lceil \frac{L(L+1)}{2} \right\rceil + 2],\ \dots, [1, 0, \dots, 0, 2^L]$$

- The root of the first sequence approaches 1.
- Roots of consecutive sequence increase at a decreasing rate.
- Root of the last sequence exceeds 2.
- Thus for $\lambda_L < 1 + \varepsilon$, roots are going up by at most ε.

□