How do Algebraic Curves Intersect?

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The Plan

1. Algebraic Curves
2. Projective Geometry
3. Intersection Multiplicity
4. Bezout's Theorem
5. The Group Law on Cubics

What are Algebraic Curves?

- An *algebraic curve* is the zero set of a polynomial in two variables. That is,

Definition

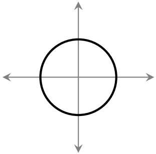
An *algebraic curve* is a set X such that X takes the form

$$\{(x, y) \mid f(x, y) = 0\}$$

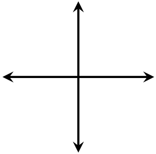
for a polynomial f in the variables x and y. Such a set is denoted by $V(f)$.

Examples of Algebraic Curves

- The unit circle $x^2 + y^2 - 1 = 0$.



- Two intersecting lines: $xy = 0$.



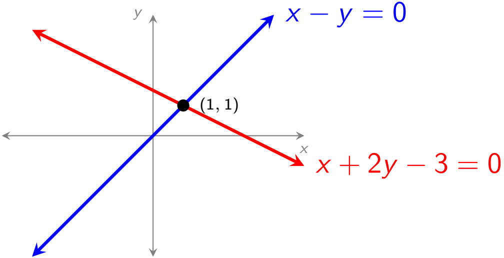
Intersections of Algebraic Curves

- One question we can ask about algebraic curves is when two of them intersect.
- For example, consider the lines $x - y = 0$ and $x + 2y - 3 = 0$.
- There are two ways we can find the intersection:

Algebra

$$\begin{cases} x - y = 0 \\ x + 2y - 3 = 0 \end{cases}$$

Picture



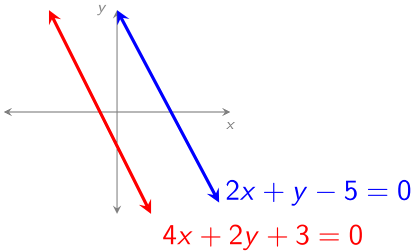
- Either way, we find that there is one intersection point: $(1, 1)$.

- Another example: consider the lines $2x + y - 5 = 0$ and $4x + 2y + 3 = 0$.
- We could try the same methods as before:

Algebra

$$\begin{cases} 2x + y - 5 = 0 \\ 4x + 2y + 3 = 0 \end{cases}$$

Picture



- This time, we don't get any intersection because the lines are parallel.

Intersections of Algebraic Curves

- Every pair of lines falls into exactly one of the following two categories:
  1. The lines intersect at exactly one point.
  2. The lines are parallel.
- We need a way to treat these cases the same way.
- In other words, a sense in which parallel lines intersect exactly once.

Projective Geometry

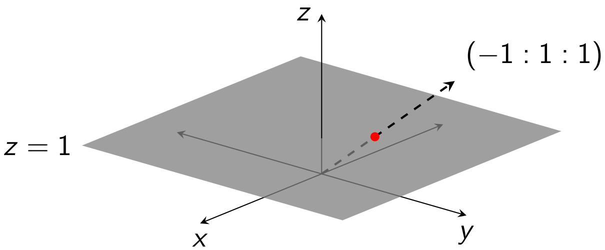
Definition

Projective 2-space, denoted $\mathbb{P}^2$, is defined as all triples of *homogeneous coordinates* $(X : Y : Z)$

- Homogeneous coordinates are triples representing a line through the origin in 3 dimensions. So $(X : Y : Z)$ and $(tX : tY : tZ)$ represent the same point.
- The convention is to differentiate between affine and projective objects by using lowercase and uppercase letters, respectively.

The Plane as a Subset of $\mathbb{P}^2$

- Consider the collection of points $(X : Y : 1)$ in $\mathbb{P}^2$. This is exactly a copy of the affine plane.



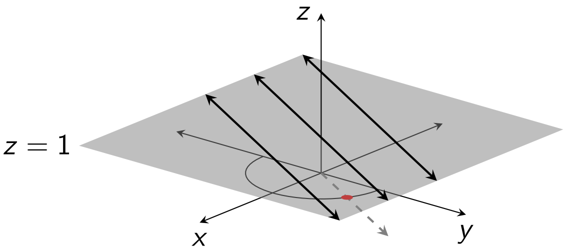
- Here the homogeneous point $(-1 : 1 : 1)$ corresponds to the affine point $(-1, 1)$.
- This works because lines through the origin can pass through the plane $z = 1$ exactly one time.

Points With $Z = 0$ as Directions

- Every other point in $\mathbb{P}^2$ has $Z = 0$, otherwise

$$(X : Y : Z) = (X/Z : Y/Z : 1)$$

- These points with $Z = 0$ are lines through the origin in the plane $z = 0$. So we can think of them as *directions* on the affine plane living in $\mathbb{P}^2$.



- These points are often called *points at infinity*.

Homogeneous Polynomials

- We know how to go between coordinates – given (x, y) in the plane, the corresponding point in $\mathbb{P}^2$ is $(X : Y : 1)$. In fact, we can make a similar construction for polynomials.
- Take a polynomial in x and y and "fill in the gaps" to create a homogeneous polynomial in X, Y, and Z.
- For example,

$$x^2 + y - 2 \rightsquigarrow X^2 + YZ - 2Z^2$$
$$x^5y - 3xy^3 + 47y \rightsquigarrow X^5Y - 3XY^3Z^2 + 47YZ^5$$

- This allows us to use projective geometry to discuss algebraic curves.

Intersections of Homogeneous Polynomials

- Now that we have homogeneous polynomials, we can ask about how they intersect and how this differs from the polynomials we had before.
- First, it is important to distinguish the two different ways for homogeneous polynomials to share a root.
  1. The root has the form $(X : Y : 1)$, in which case (x, y) is a shared root of the corresponding affine polynomial.
  2. The root has the form $(X : Y : 0)$. We need to think a little bit more about what this means.

Intersections of Homogeneous Polynomials (with $Z = 0$)

- Let's go back to the example of the parallel lines from before.
- Recall that the lines were defined by

$$2x + y - 5 = 0 \qquad \text{and} \qquad 4x + 2y + 3 = 0$$

- The corresponding homogeneous polynomials are

$$2X + Y - 5Z = 0 \qquad \text{and} \qquad 4X + 2Y + 3Z = 0$$

- We know that these don't have any shared points with $Z = 1$, because such a point would be an intersection of the two lines.
- So let's think about $Z = 0$. Then we want to find a solution to the pair of equations

$$\begin{cases} 2X + Y = 0 \\ 4X + 2Y = 0 \end{cases}$$

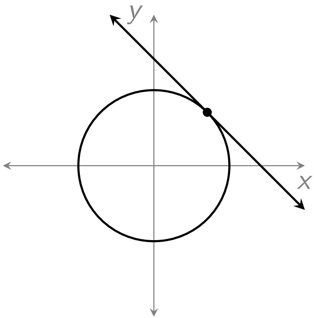
- $X = -1,\ Y = 2$ is a solution, so $(-1 : 2 : 0)$ is a shared root of our homogeneous lines.

Intersecting at Infinity

- We said before that projective points with $Z = 0$ can be thought of as *points at infinity*, or as *directions*. So what does it mean if two polynomials share one as a vanishing point?
- It means, in some sense, that the polynomials share a direction rather than a point.
- This allows us to treat the case of parallel lines the same way we treat a pair of non-parallel lines
  - The non-parallel lines share an affine point.
  - The parallel lines share a point at infinity, i.e. they share a direction.
  - This coincides nicely with the idea of being parallel, which really means that the lines are going in the same direction.

Intersection Multiplicity

- Another problem we could have is *tangent intersections*.



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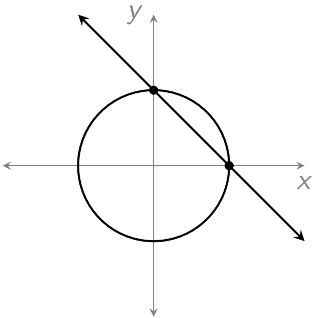


- The way we resolve this is *intersection multiplicity*. This is similar to the multiplicity of roots of polynomials.
- The more two curves look like each other at an intersection point, the higher the intersection multiplicity of that point is.

Bezout's Theorem

- We are finally ready to state Bezout's Theorem!

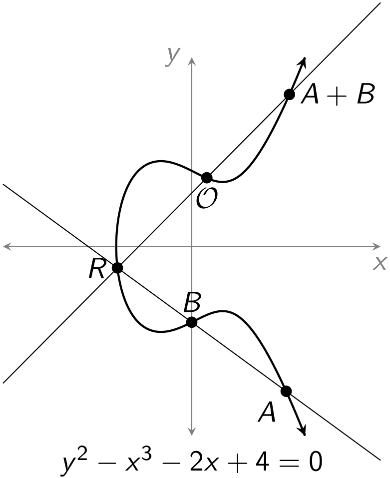
Theorem (Bezout's Theorem)

If $p(x, y)$ and $q(x, y)$ are two polynomials of respective degrees n and m, then the algebraic curves defined by $p = 0$ and $q = 0$ intersect at exactly nm points, accounting for complex roots, intersections at infinity, and intersection multiplicity.

- It is clear now that we need all the conditions in the theorem in order for it to hold:
  - If we don't allow for complex roots, some polynomials may not have any roots at all, so the corresponding algebraic curves will be empty.
  - If we don't consider intersections at infinity, some algebraic curves will not intersect.
  - If we don't consider intersection multiplicity, the number of intersections will be counted as too low.

The Group Law on Cubics

- A *cubic* is a polynomial whose largest term has degree three.



- It is convenient to take $\mathcal{O} = (0 : 1 : 0)$ because this is a vertical line.

The End

Thank you!