Math in Base-Fibonacci: Recurrences and Complete Sequences

John Haviland, joint with E. Bołdyriew, P. Lâm,
J. Lentfer, S. J. Miller, & F. Trejos Suárez

SMALL REU 2020, Williams College

Astronaut Scholar Technical Conference
August 13 – 14, 2021

Outline

Complete Sequences

Definition

A sequence (a_n) is *complete* if every positive integer can be written as a sum of its terms.

- $a_n = 2^n$, $n \geq 0$ is complete using binary expansions.
- $a_n = 3^n$ is not complete since 2 is not a sum of powers of 3.

Motivating Example

Zeckendorf's theorem implies that the Fibonacci numbers $F_1 = 1$, $F_2 = 2$, $F_n = F_{n-1} + F_{n-2}$ are complete.

- We can use complete sequences to express integers, like base-2.
- Zeckendorf's theorem lets us write in "base-Fibonacci."

Other Recurrences

Definition

A *positive linear recurrence sequence* (PLRS) is a sequence (a_n) satisfying a recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_L a_{n-L}$$

in which $c_i \geq 0$ and $c_1, c_L > 0$, with initial conditions $a_1 = 1$ and

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_{n-1} a_1 + 1$$

for $n < L$.

- A PLRS is uniquely determined by its coefficients, and so is denoted $[c_1, \ldots, c_L]$.
- PLRS naturally generalize the Fibonacci numbers.

Summary of Results

- Complete characterization for strictly positive recurrence coefficients $c_i > 0$.
- Complete characterization for $[1, \ldots, 1, 0, \ldots, 0, N]$.
- Coefficient modifications that preserve in/completeness.
  - $[c_1, \ldots, c_L]$ complete $\implies$ $[c_1, \ldots, c_L - k]$ complete.
  - $[c_1, \ldots, c_L]$ incomplete $\implies$ $[c_1, \ldots, c_L, c_{L+1}]$ incomplete.
- Analytic criteria: bounding roots of associated polynomials.
- Outside bounds, behavior of roots is chaotic.

Possibilities for Future Work

- Characterize more families of sequences.
- Finish proof of finite-time checking algorithm.
- Prove conjectured analytic bounds are correct.
- Are other relationships among coefficients useful?
    - E.g., sum of coefficients, ratio of consecutive coefficients.

References

E. Bołdyriew, J. Haviland, P. Lâm, J. Lentfer, S. J. Miller, and F. Trejos Suárez. An Introduction to Completeness of Positive Linear Recurrence Sequences, *Fibonacci Quarterly* **58** (2020), 77–90.

---. Completeness of Positive Linear Recurrence Sequences, preprint, 2021. Available `https://arxiv.org/abs/2010.01655`.

J. L. Brown. Note on complete sequences of integers, *American Mathematical Monthly* **68** (1961), 557.

T. C. Martinez, S. J. Miller, C. Mizgerd, J. Murphy, and C. Sun, Generalizing Zeckendorfs Theorem to Homogeneous Linear Recurrences II, preprint, 2020. Available `https://arxiv.org/abs/2009.07891`.

Acknowledgements

This research was conducted as part of the 2020 SMALL REU program at Williams College. This work was supported by NSF Grants DMS1947438 and DMS1561945, Williams College, Yale University, and the University of Rochester.

Thank You!!