Constructions of Generalized MSTD Sets in Higher Dimensions

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Background

Definitions

A is finite set in $\mathbb{Z}^d$, $|A|$ is its size. Define

- Sumset: $A + A = \{a_i + a_j : a_i, a_j \in A\}$.
- Difference set: $A - A = \{a_i - a_j : a_i, a_j \in A\}$.

Definition

Difference dominated: $|A - A| > |A + A|$
Balanced: $|A - A| = |A + A|$
Sum dominated (or MSTD): $|A + A| > |A - A|$.

Motivation

We often care about the sumset /difference set of $A \subseteq \mathbb{Z}$.

- Goldbach's Conjecture: $E \subseteq P + P$
- Fermat's Last Theorem: If A_n is the set of positive n-th powers, then $(A_n + A_n) \cap A_n = \emptyset$ for all $n \geq 3$

Natural question: What are the sizes of the sumsets/difference sets?

History

How big do we expect the sumset to be? How big do we expect the difference set to be?

- $x + y = y + x$ and $x - y \neq y - x$.

Conway's MSTD set: $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$

- $|A + A| = 26$
- $|A - A| = 25$

Nathanson, *Problems in Additive Number Theory*: "With the right way of counting the vast majority of sets satisfy $|A - A| > |A + A|$."

History

Martin-O'Bryant: A positive percentage of sets $A \subset [0, n-1]$ are MSTD as $n \to \infty$.

Zhao: The percentage approaches a limit and

$$\lim_{n\to\infty} \frac{\#\{A \subseteq [0, n-1];\ A \text{ is sum-dominant}\}}{2^n} > 0.000428.$$

Generalized MSTD

Constructing MSTD Sets

- Say $A \subseteq [0, n]$, then $x \in A + A$ if we can find $a_1, a_2 \in A$ such that $a_1 + a_2 = x$.
- The number of pairs in $[0, n]$ that sum to x is large, except when x is near 0 or $2n$.
- With high probability, the middle will be full, but the fringes will be missing elements
- As the fringes in the sumset and difference set are made by fringes in the original set, the trick is to control the fringes.

Definitions

We generalize the idea of sumsets and difference sets:

$$sA - dA = \underbrace{A + \cdots + A}_{s \text{ times}} - (\underbrace{A + \cdots + A}_{d \text{ times}}),$$

$$a_1 + \cdots + a_s - (a_{s+1} + \cdots + a_{s+d}) \in sA - dA.$$

Previous work by SMALL REU students showed

- For any $s_1 + d_1 = s_2 + d_2$, there exists a set A such that $|s_1 A - d_1 A| > |s_2 A - d_2 A|$
- For any $k \in \mathbb{N}$, there exists a set A such that $|cA + cA| > |cA - cA|$ for all $1 \leq c \leq k$
- There does not exist a set A such that $|kA + kA| > |kA - kA|$ for all k.

Questions

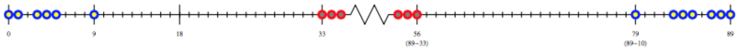
Can we extend these results to higher dimensions?

- For any $s_1 + d_1 = s_2 + d_2$, can we find a set $A \subset \mathbb{Z}^2$ such that $|s_1 A - d_1 A| > |s_2 A - d_2 A|$? Yes!
- Given $k \in \mathbb{N}$, can we find a set $A \subset \mathbb{Z}^2$ such that $|cA + cA| > |cA - cA|$ for all $1 \leq c \leq k$? Yes!
- Can we prove that there does not exist a set $A \subset \mathbb{Z}^2$ such that $|kA + kA| > |kA - kA|$ for all k? In some cases!

1-Dimensional Constructions

- How did previous SMALL students construct 1-dimensional sets such that $|s_1A + d_1A| > |s_2A - d_2A|$?
- Recall that fringes are very important, the middle is not that important.

$$L = [0, 2k+1] \setminus (\{2\} \cup [k+2, 2k])$$
$$R = [0, 2k+2] \setminus (\{3\} \cup [k+3, 2k+1])$$



- The fringes maintain their shape when added and subtracted, but after enough additions and subtractions, the middle will cover the holes in the fringes.

2-Dimensional Constructions

How do the 1-dimensional constructions generalize to 2-dimensions?

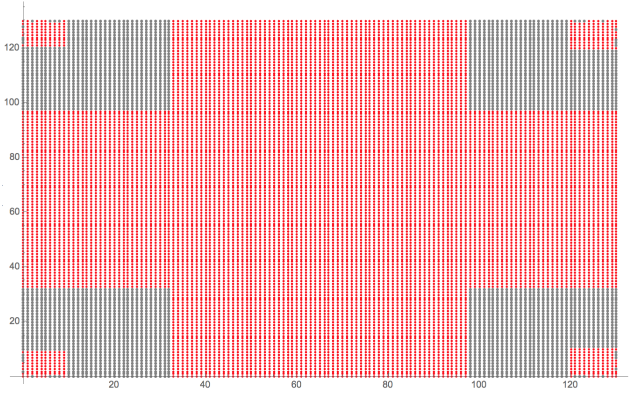


Figure: 2-dimensional generalized MSTD set

2-Dimensional Constructions

How do the 1-dimensional constructions generalize to 2-dimensions?

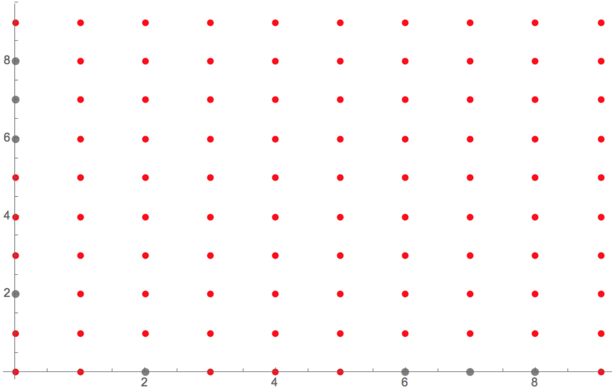


Figure: Zooming into the fringe in the corner

Generations

k-Generational Sets

- Using this construction, for $s_1 + d_1 = s_2 + d_2 = k$ we can find a set $A \subset \mathbb{Z}^2$ such that $|s_1 A - d_1 A| > |s_2 A - d_2 A|$.

- We can prove that for any $x_1 + y_1 = x_2 + y_2 \neq k$, we have $|x_1 A - y_1 A| = |x_2 A - y_2 A|$.

- We can then use these sets to create a set $A' \subset \mathbb{Z}^2$ such that $|cA' + cA'| > |cA' - cA'|$ for all $1 \leq c \leq k$. These sets are known as k-*generational*.

- To construct k-generational sets, we will need to introduce *base expansion*.

Base Expansion

Idea behind base expansion:

- For sets $A, B \subset \mathbb{Z}^2$ and $m \in \mathbb{N}$ sufficiently large (relative to A) we define:
$$C = m \cdot A + B$$
- We have proved
$$|sC - dC| = |sA - dA| \cdot |sB - dB| \,.$$

k-Generational Existence

Recall: A set A such that $|cA + cA| > |cA - cA|$ for all $1 \leq c \leq k$ is k*-generational*.

For each i, choose A_i with $|iA_i + iA_i| > |iA_i - iA_i|$ and $|jA_i + jA_i| = |jA_i - jA_i|$.

Define $A = A_1 + mA_2 + m^2A_3 + \cdots + m^{k-1}A_k$.

k-Generational Existence

Define $A = A_1 + mA_2 + m^2A_3 + \cdots + m^{k-1}A_k$.

$$\begin{aligned}
|jA + jA| &= \prod_{i=1}^{k} |jA_i + jA_i| \\
&= |jA_j + jA_j| \cdot \prod_{i \neq j} |jA_i + jA_i| \\
&= |jA_j + jA_j| \cdot \prod_{i \neq j} |jA_i - jA_i| \\
&> |jA_j - jA_j| \cdot \prod_{i \neq j} |jA_i - jA_i| \\
&= |jA - jA| .
\end{aligned}$$

Limiting Behavior of kA

Are there any 2-dimensional sets such that $|kA + kA| > |kA - kA|$ for all $k \in \mathbb{N}$?

First we have to describe the behavior of kA.

Theorem (Nathanson)

Let $A = \{a_0, a_1, \ldots, a_k\}$ be a finite set of integers with $a_0 = 0 < a_1 < \cdots < a_m = a$ and $(a_1, a_2, \ldots, a_m) = 1$. Then there exists non-negative integers c and d and sets $C \subset [0, c-2]$ and $D \subset [0, d-2]$ such that for all $k \geq a^2 m$,

$$kA \ = \ C \cup [c, ka - d] \cup ka - D$$

Limiting Behavior of kA

Theorem

Let $A \subset \mathbb{Z}^2$. Let a and b be the smallest non-zero x and y coordinates, a' and b' be the largest x and y coordinates, and $N = \max\{2a'^2, 2b'^2\}$. If $(a, a') = 0$, $(b, b') = 0$, and $\{(0,0), (a,0), (0,b), (a',0), (0,b'), (a,b), (a,b'), (a',b), (a',b')\} \subset A$, then for $k \geq N$ and for some constants C, c_1, c_2, we have $|kA| = k^2 a' b' - C - c_1 k - c_2 k$.

Limiting Behavior of kA

We want to show for sufficiently large k, the amount of elements missing from kA grows linearly.

$|kA - kA| \geq |kA + kA|$

Theorem

Let $A \subset \mathbb{Z}^2$. Let a and b be the smallest non-zero x and y coordinates, a' and b' be the largest x and y coordinates, and $N = \max\{2a'^2, 2b'^2\}$. If $(a, a') = 0$, $(b, b') = 0$, and $\{(0,0),(a,0),(0,b),(a',0),(0,b'),(a,b),(a,b'),(a',b),(a',b')\} \subset A$, then for $k \geq N$, we have $|kA - kA| \geq |kA + kA|$.

$|kA - kA| \geq |kA + kA|$

We want to show $|kA - kA| \geq |kA + kA|$.

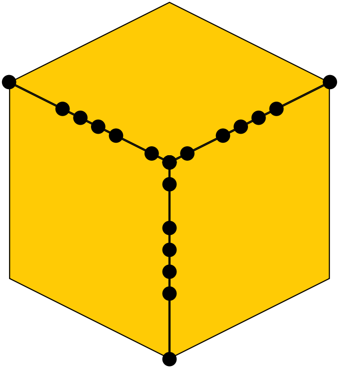
Other Constructions

d-Dimensional Constructions

What does the middle look like in d-dimensions?

d-Dimensional Constructions

What do the fringes look like in *d*-dimensions?



Other 2-Dimensional Constructions

Needs to have integer vertices and be locally point symmetric.

Parallelogram with slope m.

Define $\varphi : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $\varphi(x, y) = (x + my, y)$.

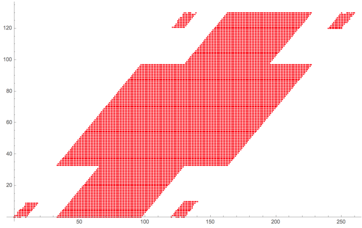


Figure: The generalized MSTD set for $k = 4$, $n = 130$, $s_1 = 4$, $d_1 = 0$, $s_2 = 2$, and $d_2 = 2$ that has been sheared with slope $m = 1$

Parallelogram d-Dimensional Constructions

- $d(d-1)/2$ positive directions to shear the set
- $d(d-1)/2$ slopes:

$$m_{1,2}, m_{1,2}, \dots, m_{1,d}, m_{2,3}, \dots, m_{2,d}, \dots, m_{d-1,d}$$

($m_{i,j}$ is the jth axis sheared in the ith direction)

- We define $\psi : \mathbb{Z}^2 \to \mathbb{Z}^2 d$ by

$$\begin{aligned}\psi(x_1, x_2, \dots, x_d) \;=\; &(x_1 + m_{1,2}x_2 + m_{1,3}x_3 + \dots + m_{1,d}x_d, \\ &x_2 + m_{2,3}x_3 + \dots + m_{2,d}x_d, \dots, x_d).\end{aligned}$$

Conclusion

Future Directions

- We have shown the elements missing from kA grows linearly for certain A
- **In the future:** show that the elements missing from kA grows linearly for all A
- Previous work in 1-dimensions has shown positive percentages for generalized MSTD sets, chains of generalized MSTD sets, and k-generational sets
- **In the future:** Show positive percentages for d-dimensional sets

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