Analysis on Fractals

From *Differential Equations on Fractals*, by Robert Strichartz

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Fractals: Their Beauty and Topology
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What's Ahead

Self-Similar Identities

- Contraction maps, words, and composition
  - What does a contraction map do?
  - Composing functions
- The dyadic points as a dense set
  - What is density in math?
  - How do we know the dyadic points are dense?
  - Continuous functions with a dense set
- Constructing the Sierpinski Gasket with contraction maps
  - Extending ideas from the Interval
  - Start with 3 points instead

Structure on Self-Similar Fractals

- Cell structure on the Interval and Gasket
  - Start with entire shape instead of individual points
- Graphs and topological structure
  - "Neighbors" of points
  - What changes at the boundary points?

Measure

- Let K be a self-similar set and C be any cell in K. Then a *measure*, denoted μ, on K fulfills four properties:
  1. **Positivity:** $\mu(C) > 0$
  2. **Additivity:** if C is the union of some cells $C_1, C_2, \ldots, C_m$ and all C_j intersect only at boundary points, then

$$\mu(C) = \sum_{j=1}^{m} \mu(C_j)$$

  3. **Continuity:** as the size of $C \to 0$, $\mu(C) \to 0$.
     In other words, the measure of a point is always 0.
  4. **Probability:** $\mu(K) = 1$.
     For example, for all measures μ on the Interval, $\mu(I) = 1$, and likewise for the Gasket.

Measure

- The symbol μ_w means $\mu(F_w(K))$, the measure of the cell given the word w.
- If C is a cell given by $F_w(K)$, where $|w| = m$, then we can express the measure of C as a product:

$$\mu(C) = \prod_{j=0}^{m} \mu_{w_j}$$

Definitions for Integration

- When working with fractals like the Interval and the Gasket, we take integrals with respect to a measure.

Definition

$$\int_K f \; d\mu = \lim_{m\to\infty} \sum_{|w|=m} f(x_w)\mu_w$$

- Another definition can be used more easily to compute integrals:

Definition

$$\int_K f \; d\mu = \sum_i \mu_i \int_K f \circ F_i \; d\mu$$

Some Examples

- Functions take a point in K to another point in Euclidean space.
- If $f(x) = x$, then $\int_{SG} x \, d\mu = \sum_{i=0}^{2} \mu_i \int_{SG} F_i \, d\mu$
- Extending to $f(x) = x^n$
  - The binomial theorem: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$
- Extending to any polynomial function
- Implementation in Python
  - bit.ly/si-integration
  - Accepts any starting points, measure, and polynomial function
- Extending to arbitrary functions with Taylor series

Defining Energy

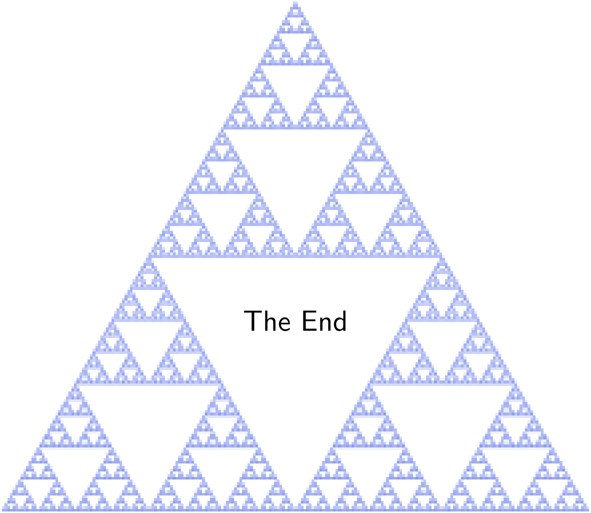
Definition

For a finite, connected graph G and real-valued function u, the *graph energy* is defined by

$$E_G(u) = \sum_{x \sim y} (u(x) - u(y))^2$$

Properties of Graph Energy

- Polariztion Identity
- **Markov Property**: if u is replaced by a minimum or maximum value and a constant, then energy reaches a limit and can no longer increase, because each term in the total sum is either staying constant or decreasing.
- The **$1/5$ - $2/5$ Rule** states that the value at any inside point is a weighted average of the boundary point values. This works for the Interval as well as the Gasket.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