Derived Categories

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Abstract

We undertake a fundamental construction in homological algebra: the derived category of an arbitrary abelian category. After translating familiar homological notions like kernels, exactness, and cohomology into categorical terms, we construct the derived category via localization and discuss a case where localization is not necessary for understanding the derived category.

Contents

1 Introduction

Derived categories are of central importance in homological algebra. Understanding how functors fail to be exact can be extremely helpful for computing invariants. As an easy example, knowing that every finite-dimensional vector space is flat (i.e., the tensor product is exact on finite-dimensional vector spaces) gives us a way to inductively compute the dimension of a tensor product. Since we have the split exact sequence

$$0 \to k^{n-1} \to k^n \to k \to 0, \tag{1.1}$$

if we tensor with another vector space V, we obtain

$$0 \to k^{n-1} \otimes V \to k^n \otimes V \to V \to 0. \tag{1.2}$$

This sequence also splits (every short exact sequence of vector spaces splits), so by induction it follows that $V \otimes k^n = n \dim V$. In more interesting settings, the situation is much more complicated, but exact sequences are still a powerful computational tool and it is valuable to measure how exactness holds or fails.

A functors (left or right) derived functors are usually how we measure exactness. Derived categories, by replacing objects with complexes and considering equivalence up to cohomology rather than requiring strict equivalence, provide a natural setting for derived functors. Although derived functors can be defined without derived categories, their construction is more natural in the context of derived categories.

Our presentation roughly follows that of Manin and Gelfand's *Methods of Homological Algebra* [2], with some deviations in order and details. Weibel's *An Introduction to Homological Algebra* [3] is also an excellent reference and offers more content on other aspects of the subject.

In Section 2, we show how useful features of the categories Ab and, more generally, Mod_R can be reformulated categorically. This includes the additive structure inherited by maps and objects as well as the canonical decomposition of a map into a sequence of kernels and cokernels.

In Section 3, we use the categorical constructions of abelian categories to define familiar homological constructions like cohomology and chain homotopies. We see how a cochain map induces a map in each degree of homology and that this makes cohomology a functor from complexes to objects. We also discuss how homotopies give a weaker notion of equivalence of maps and homotopic maps induce the same maps in cohomology. This means we have a category of complexes and homotopy classes of chain maps, which is interchangable with the usual category of complexes in the construction of the derived category.

Finally, in Section 4, we construct the derived category of an abelian category by adjoining formal inverses of quasi-isomorphisms to the category of complexes. We go on to show that localization is a necessary process and in particular, using cohomology to form a complex does not give the same result in general.

Because it is fixed in the literature, we will take the cochain/cohomological convention that complexes are increasing in degree. We will also use common notational conveniences like writing fg for the composition $f \circ g$ and writing $X \in \mathcal{C}$ to mean X is an object in the category $\mathcal{C}$ even though $X \in \operatorname{Obj} \mathcal{C}$ or something similar would be more precise.

2 Additive and Abelian Categories

In this section we cover the necessary facts about additive and abelian categories for our construction and discussion of derived categories.

2.1 Additive Categories

As the name suggests, the key features of additive categories are ways to add things. More specifically, ways to add maps and objects. Note that Ab and Mod_R, we add maps by elementwise and add objects by taking direct sums, which (at least in the finite case) are simultaneously products and coproducts. In a categorical setting, adding maps elementwise doesn't make sense, so we will explicitly require that there is a way to add maps in an additive category.

Definition 2.1 (Pre-additive category)**.** A category $\mathcal{C}$ is *pre-additive* if $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ has an abelian group structure for all $X, Y \in \mathcal{C}$ and composition distributes over addition on both sides.

This additive structure on $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ turns out to be helpful when thinking about how to add objects.

Proposition 2.2. *If $\mathcal{C}$ is a pre-additive category, then the product of a finite collection of objects (if it exists) is also their coproduct, and vice-versa.*

Proof. For some objects $X_1, \dots, X_n \in \mathcal{C}$, define $\delta_{ij} \colon X_i \to X_j$ by

$$\delta_{ij} = \begin{cases} 0 \in \operatorname{Hom}_{\mathcal{C}}(X_i, X_j) & \text{if } i \neq j \\ \mathrm{id}_{X_j} & \text{if } i = j. \end{cases} \tag{2.1}$$

If $\prod_{j=1}^n X_j$ exists in $\mathcal{C}$, then for each i, the maps δ_{ij} define a map $X_i \to \prod_{j=1}^n X_j$ by the universal property of the product and these maps make $\prod_{j=1}^n X_j$ satisfy the universal property for the coproduct of the X_j's. If $\coprod_{i=1}^n X_i$ exists in $\mathcal{C}$, then for each j, the maps δ_{ij} make $\coprod_{i=1}^n X_i$ the product of the X_i's. □

In a pre-additive category, if the product (equivalently, coproduct) of a finite collection of objects $X_1, \dots, X_n$ exists, we call it the *direct sum* and denote it by $\bigoplus_{i=1}^n X_i$. It is equipped with canonical maps $X_j \to \bigoplus_{i=1}^n X_i$ and $\bigoplus_{i=1}^n X_i \to X_j$ for all j. As a special case, the direct sum of two objects $X, Y \in \mathcal{C}$ (if it exists) is denoted by $X \oplus Y$.

This is our notion of adding objects. So we define an additive category to be one where we can add maps (pre-additive) and we can always add objects (all direct sums exist).

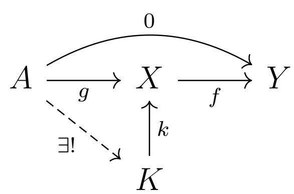
Definition 2.3 (Additive category)**.** A category is *additive* if it is pre-additive and all finite direct sums exist.

Remark 2.4. A initial object is the coproduct of an empty collection of objects, while a terminal object is the product of an empty collection. Thus, an additive category has an object which is both initial and terminal, i.e., a zero object. An important property is that the (categorical) zero map $X \to 0 \to Y$ is the (additive) zero element of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$; this follows from the fact that composing with the (additive) zero map gives zero (since $a0 = a0 + a0$ and $0b = 0b + 0b$) and $\mathrm{Hom}_{\mathcal{C}}(X, 0) = \mathrm{Hom}_{\mathcal{C}}(0, Y) = 0$ as abelian groups.

2.2 Kernels and Cokernels; Canonical Decompositions

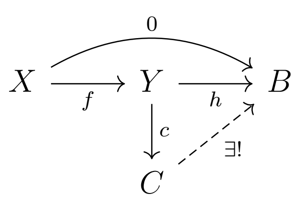
In any category with a zero, we can define kernels and cokernels of maps.

Definition 2.5 (Kernel, cokernel)**.** Suppose $\mathcal{C}$ is a category with a 0 object. A map $k\colon K \to X$ is the *kernel* of the map $f\colon X \to Y$ if, for any $g\colon A \to X$ such that $fg = 0$, g factors uniquely through k.



(2.2)

Dually, a map $c\colon Y \to C$ is the *cokernel* of the map $f\colon X \to Y$ if, for any $h\colon Y \to B$ such that $hf = 0$, h factors uniquely through c.



(2.3)

The kernel of f is denoted by $\ker f$ and the cokernel by $\operatorname{cok} f$. We will frequently refer to the kernel and cokernel by the objects K and C rather than the maps k and c.

Thinking in terms of Ab and Mod_R, the map $A \to \ker f$ as in (2.2) is the inclusion of the image of g into $\ker f$; such an inclusion must exist becuase $fg = 0$. In (2.3), the kernel of h must include $\operatorname{im} f$ because $hf = 0$, so the map $C \to B$ is the projection of $C = Y/\operatorname{im} f$ onto $B = Y/\ker h$.

Proposition 2.6. *The following are important properties of kernels and cokernels in an additive category:*

- *Every kernel in an additive category is a monomorphism; every cokernel is an epimorphism.*

- *A map* $f\colon X \to Y$ *is monic if and only if* $\ker f = 0$ *and epic if and only if* $\operatorname{cok} f = 0$. □

If the kernel and cokernel of a map $f\colon X \to Y$ both exist, then we have the decomposition

$$\ker f \longrightarrow X \xrightarrow{f} Y \longrightarrow \operatorname{cok} f. \tag{2.4}$$

The kernel and cokernel are nice maps (in particular, monic and epic), but f may be very strange. The following decompositions are nicer, but there may not be a way to connect them.

$$\ker f \longrightarrow X \longrightarrow \operatorname{cok}(\ker f) \qquad \text{and} \qquad \ker(\operatorname{cok} f) \longrightarrow Y \longrightarrow \operatorname{cok} f \tag{2.5}$$

In Ab and Mod_R, we can always connect these sequences: we can decompose any map $f\colon X \to Y$ as

$$\ker f \longrightarrow X \xrightarrow{f} \operatorname{im} f \hookrightarrow Y \longrightarrow \operatorname{cok} f, \tag{2.6}$$

where $\operatorname{im} f = \operatorname{cok}(\ker f) = \ker(\operatorname{cok} f)$. This is called the *canonical decomposition* of f. In abelian categories, we want to be able to similarly decompose any map, so the existence of such a decomposition

Definition 2.7 (Abelian category)**.** A category is *abelian* if it is additive, every map has both a kernel and cokernel, and if f is any map, then $\operatorname{cok}(\ker f)$ and $\ker(\operatorname{cok} f)$ are canonically isomorphic.

This definition allows us to unify the separate decompositions in (2.5), so every map has a canonical decomposition as in (2.6). Motivated by this definition, we denote $\operatorname{cok}(\ker f) = \ker(\operatorname{cok} f)$ by $\operatorname{im} f$. Note that in more general settings, we can use $\operatorname{im} f$ to refer to $\ker(\operatorname{cok} f)$ and $\operatorname{coim} f$ to refer to $\operatorname{cok}(\ker f)$.

Another convenient consequence of Definition 2.7 is that every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel; in particular, a map is monic if and only if it is a kernel and epic if and only if it is a cokernel. In fact, this property can replace the existence of the canonical decomposition. Aluffi [1] uses this definition and proves the existence of the canonical decomposition in chapter IX.

3 Complexes

Since we want to work with complexes when constructing the derived category, we need to figure out how to emulate the usual homological constructions with the formalism of Section 2.

3.1 The Category of Complexes

Defining complexes is easy enough and can be done in additive cateogories, and we can define cohomology by examining kernels and cokernels in abelian categories.

Definition 3.1 (Complex, cohomology, exact)**.** (a) A *complex* in an additive category $\mathcal{C}$ is a sequence of objects C^n and maps $d^n\colon C^n \to C^{n+1}$, $n \in \mathbb{Z}$, such that $d^{n+1} \circ d^n = 0$. The maps d^n are called *boundary* or *differential* maps and usually written without indexes, so their defining condition is $d \circ d = 0$. We denote such a complex by $C^\bullet$.

$$C^\bullet\colon \cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots \tag{3.1}$$

(b) If $\mathcal{C}$ is abelian, then we can factor d^n through $\ker d^{n+1}$ by a^n and d^{n+1} through $\operatorname{cok} d^n$ by b^{n+1} using the universal properties. We then define $H^n(C^\bullet) = \operatorname{cok} a^n = \ker b^n$.

$$\begin{array}{ccccc}
 & H^n(C^\bullet) := \ker b^{n+1} \longrightarrow & \operatorname{cok} d^n & \searrow^{b^{n+1}} & \\
 & & \downarrow & & \\
C^n & \xrightarrow{d^n} & C^{n+1} & \xrightarrow{d^{n+1}} & C^{n+2} \\
 & \searrow_{a^n} & \uparrow & & \\
 & & \ker d^{n+1} & \longrightarrow & \operatorname{cok} a^n =: H^n(C^\bullet)
\end{array} \tag{3.2}$$

We say $C^\bullet$ is *exact at* C^n if $H^n(C^\bullet) = 0$ and simply *exact* if it is exact at each term.

To check that this definition of cohomology matches the usual definition in Ab and Mod_R, note that $\operatorname{cok} a^n$ is (in this case) the quotient of $\ker d^{n+1}$ by the image of a^n, but a^n is just the inclusion of $\operatorname{im} d^n$ into $\ker d^{n+1}$, so we recover the usual $H^n(C^\bullet) = \ker d^{n+1}/\operatorname{im} d^n$. For the other definition, $H^n(C^\bullet) = \ker b^{n+1}$, note that $\operatorname{cok} d^n = C^{n+1}/\operatorname{im} d^n$, and b^n is induced from d^{n+1}, so $\ker b^n$ is the elements the projected to $C^{n+1}/\operatorname{im} d^n$ from $\ker d^{n+1}$. That is, $\ker b^n = \ker d^{n+1}/\operatorname{im} d^n$.

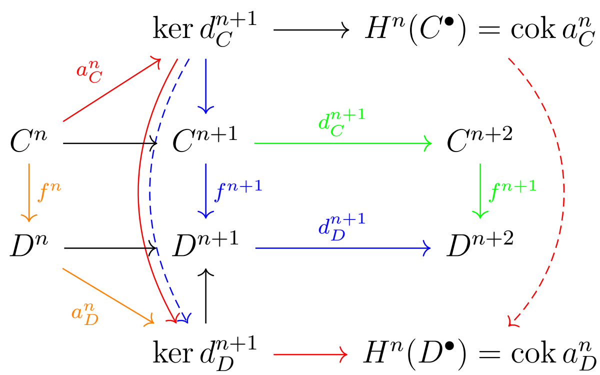
Of course, this does not prove that $\ker b^{n+1} \cong \operatorname{cok} a^n$ in any abelian category, but since we have plenty more to discuss about complexes, the intuitive explanation will have to suffice. Our next step is to define the category of complexes over an abelian category and show that each degree of cohomology gives a functor from complexes to spaces.

Definition 3.2 (Map of complexes)**.** If $C^\bullet, D^\bullet$ are complexes in an additive category, then a *map of complexes* or *cochain map* $f\colon C^\bullet \to D^\bullet$ is a sequence of maps $f^n\colon C^n \to D^n$ that commute with the boundary maps. More precisely, the following diagram commutes for all n.

$$\begin{array}{ccccccccc} C^\bullet\colon & \cdots & \longrightarrow & C^n & \xrightarrow{d^n} & C^{n+1} & \longrightarrow & \cdots \\ & & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ D^\bullet\colon & \cdots & \longrightarrow & D^n & \xrightarrow{d^n} & D^{n+1} & \longrightarrow & \cdots \end{array} \tag{3.3}$$

It is clear that if we have two cochain maps, then their composition, defined by composition in each degree, is also a cochain map. Thus, we have the *category of complexes in* $\mathcal{C}$, denoted $\operatorname{Kom}\mathcal{C}$. As noted above, taking cohomology defines a functor $\operatorname{Kom}\mathcal{C} \to \mathcal{C}$. More precisely, for each n, we have defined $H^n\colon \operatorname{Kom}\mathcal{C} \to \mathcal{C}$ on objects, but we need to know how to define it on maps.

Suppose $f\colon C^\bullet \to D^\bullet$ is a cochain map. The following diagram summarizes our argument.



(3.4)

We follow the blue arrows, then the red arrows. The blue composition $\ker d_C^{n+1} \to D^{n+2}$ is zero because we can apply commutativity to use the green arrows instead, and the blue/green composition involves $\ker d_C^{n+1} \to C^{n+1} \to C^{n+2}$, hence is zero. By the kernel universal property applied to $\ker d_D^{n+1}$, this gives the dashed blue map $\ker d_C^{n+1} \to \ker d_D^{n+1}$.

Now, the red composition (the map $\ker d_C^{n+1} \to \ker d_D^{n+1}$ is the one we just constructed; it is repeated as it needs to be more than one color) is zero because we can apply commutativity to use the orange arrows instead, and the orange/red composition involves $D^n \to \ker d_D^{n+1} \to \operatorname{cok} a_D^n$, hence is zero. By the cokernel universal property applied to $H^n(C^\bullet) = \operatorname{cok} a_C^n$, this gives the dashed red map $H^n(C^\bullet) \to H^n(D^\bullet)$ as desired.

Examining (3.4), we can imagine adding another complex $E^\bullet$ under the diagram with a cochain map $g\colon D^\bullet \to E^\bullet$. Then commutativity of the huge diagram would tell us that the composition $H^n(g)H^n(f)$ we get from chasing f and g separately is the same as the map $H^n(gf)$ we get from chasing gf. Thus, cohomology is a functor $\operatorname{Kom}\mathcal{C} \to \mathcal{C}$.

Henceforth, we will omit indices from out maps to avoid cluttered notation and diagrams. This does not cause any real problems because the relevant indices can always be deduced from how maps are being composed and compared.

3.2 Homotopy

Homotopies give us a weaker sense of equivalence between cochain maps.

Definition 3.3 (Homotopy)**.** If $f, g \colon C^\bullet \to D^\bullet$ are two cochain map, a *homotopy* between f and g is a sequence of maps $h \colon C^{n+1} \to D^n$ such that $f - g = hd + dh$.

$$\begin{array}{ccccccccc} C^\bullet\colon \cdots & \longrightarrow & C^{n-1} & \xrightarrow{d} & C^n & \xrightarrow{d} & C^{n+1} & \longrightarrow & \cdots \\ & & g\downarrow\downarrow f & \swarrow^{h} & g\downarrow\downarrow f & \swarrow^{h} & g\downarrow\downarrow f & & \\ D^\bullet\colon \cdots & \longrightarrow & D^{n-1} & \longrightarrow & D^n & \longrightarrow & D^{n+1} & \longrightarrow & \cdots \end{array} \tag{3.5}$$

Note that this diagram *does not* commute, and h is *not* a map of complexes. If such a homotopy exists between f and g, we say f and g are *homotopic* and write $f \simeq g$.

Remark 3.4. The name *homotopy* comes from algebraic topology, in which a homotopy is a way to continuously deform a map of spaces into another map. In this setting, homotopies as we have defined them are called *chain homotopies* to avoid ambiguity. The connection between the two notions is that topological homotopies between maps of spaces induce chain homotopies betwen the induced maps on singular chains, and hence the same map in singular homology. This implies homotopy equivalences induce isomorphisms in singular homology.

It is easy to see that $\simeq$ is an equivalence relation on maps in $\mathrm{Kom}\,\mathcal{C}$ and that compositions of homtopic maps are homotopic (admittedly, this is less clear, but amounts to a diagram chase), so we can define a new category $\mathrm{K}\,\mathcal{C}$ whose objects are complexes in $\mathcal{C}$ and whose maps are homotopy classes (i.e., equivalence classes of $\simeq$) of maps in $\mathrm{Kom}\,\mathcal{C}$.

A very important property of homotopies is that homotopic cochain maps induce the same maps in cohomology. This becomes relevant when we look at derived categories, which turn equivalences in cohomology into actual equivalences, so we will only need to think about cochain maps up to homotopy.

Proposition 3.5. *If $f, g \colon C^\bullet \to D^\bullet$ are homotopic cochain maps, then for all n, $H^n(f) = H^n(g)$.*

Proof. It suffices to show that if $f \simeq 0$, then $H^n(f) = 0$. This amounts to a diagram chase by expressing $H^n(f) = H^n(hd) + H^n(dh)$ for some homotopy h showing $f \simeq 0$ and using the diagram from (3.4). We have spent enough time diagram chasing for this section, so we omit the details. ☐

4 Constructing the Derived Category

4.1 Localization of a Category

The construction of the derived category is a generalization of the construction of the localization of a ring. We want to be able to look at a category of complexes, but under a weaker notion of equivalence than cochain isomorphisms. In particular, two complexes should be considered the same if they have the same cohomology. The easiest way to do this is to simply declare that any cochain maps which descends to an isomorphisms in every degree of cohomology are isomorphisms, and see what results.

Definition 4.1 (Quasi-isomorphism)**.** A cochain map $f\colon C^\bullet \to D^\bullet$ is a *quasi-isomorphism* if $H^n(f)$ is an isomorphism for all n. In this case, we say $C^\bullet$ is *quasi-isomorphic* to $D^\bullet$.

Thus, by Proposition 3.5, any map homotopic to a cochain isomorphism is a quasi-isomorphism. Note that *a priori*, being quasi-isomoprhic is not a symmetric relation, since a quasi-isomorphism does not need to have an inverse. Similarly, two complexes with equal homology need not be quasi-isomorphic.

We define the derived category of an abelian category by essentially the same universal property used for the localization of a ring. This strengthens the idea that we are formally adjoining inverses to quasi-isomorphisms.

Theorem 4.2. *Suppose $\mathcal{C}$ is an abelian category. There is a category $D\mathcal{C}$ with a functor $Q\colon \operatorname{Kom}\mathcal{C} \to D\mathcal{C}$ such that*

(1) for any quasi-isomorphism f in $\operatorname{Kom}\mathcal{C}$, $Q(f)$ is an isomorphism in $D\mathcal{C}$;

(2) if $F\colon \operatorname{Kom}\mathcal{C} \to \mathcal{B}$ is a functor taking quasi-isomorphisms to isomorphisms, then F factors uniquely through Q in the sense that there is a unique functor $G\colon D\mathcal{C} \to \mathcal{B}$ such that $F = GQ$.

We call $D\mathcal{C}$ the derived category *of $\mathcal{C}$.*

The first condition ensures that every quasi-isomorphism becomes an isomorphism in $D\mathcal{C}$, while the second ensures that $D\mathcal{C}$ is the smallest possible category containing inverses for all quasi-isomorphisms.

Proof. We will work in much more generality: suppose instead that $\mathcal{A}$ is an arbitrary category and $\mathcal{S}$ is a collection of maps in $\mathcal{A}$. We construct a category $\mathcal{S}^{-1}\mathcal{A}$ and a "localization functor" $Q\colon \mathcal{A} \to \mathcal{S}^{-1}\mathcal{A}$ as follows: to start, the objects of $\mathcal{S}^{-1}\mathcal{A}$ are the objects of $\mathcal{A}$.

For the maps of $\mathcal{S}^{-1}\mathcal{A}$, construct a direct graph whose vertices are the objects of $\mathcal{A}$ and whose edges are the maps of $\mathcal{A}$, oriented from source to target, along with a reversed edge for each element of $\mathcal{S}$. So if $f\colon X \to Y$ is a map in $\mathcal{A}$, then $X \xrightarrow{f} Y$ is an edge in the graph and if $f \in \mathcal{S}$, there is also a corresponding edge $Y \to X$, which we label x_f. A map in $\mathcal{S}^{-1}\mathcal{A}$ is a path in this graph, up to replacing

$$X \xrightarrow{f} Y \xrightarrow{g} Z \qquad \text{with} \qquad X \xrightarrow{gf} Z \tag{4.1}$$

and

$$X \xrightarrow{f} Y \xrightarrow{x_f} X \quad \text{with} \quad X \xrightarrow{\text{id}} X \qquad \text{or} \qquad Y \xrightarrow{x_f} X \xrightarrow{f} Y \quad \text{with} \quad Y \xrightarrow{\text{id}} Y \tag{4.2}$$

for any $f\colon X \to Y$, $g\colon Y \to Z$ in $\mathcal{A}$. The composition of two maps is defined by concatenation.

Finally, the functor $Q\colon \mathcal{A} \to \mathcal{S}^{-1}\mathcal{A}$ takes X to itself and a map $f\colon X \to Y$ to the path $X \xrightarrow{f} Y$ (which has length 1). If $F\colon \mathcal{A} \to \mathcal{B}$ takes elements of $\mathcal{S}$ to isomorphisms, define $G\colon \mathcal{S}^{-1}\mathcal{A} \to \mathcal{B}$ on objects by $G(X) = F(X)$ and on maps by $G(f) = F(f)$, $G(x_f) = F(f)^{-1}$, and the obeying functoriality. Then clearly $F = GQ$ and every definition we made for G is forced by having $F = GQ$.

Finally, we can now take $\mathcal{A} = \operatorname{Kom}\mathcal{C}$ and $\mathcal{Q}$ to be the set of quasi-isomorphisms, and define $D\mathcal{C} = \mathcal{Q}^{-1}\mathcal{A}$. ☐

Example 4.3. If R is a ring, we can think of R as a pre-additive category with one object $*$ and $\operatorname{Hom}(*, *) = \operatorname{End}(R)$. Then if $S \subseteq R$, we can also think of S as a set of maps, and localizing R at S in the usual sense is the same as localizing the category R at the set of maps S.

4.2 Splitting

The localization construction seems like a lot of work for our goal, which was to make quasi-isomorphisms into isomorphisms. After all, cohomology can give us a functor $H\colon \operatorname{Kom}\mathcal{C} \to \operatorname{Kom}\mathcal{C}$ by sending $H(C^n, d) = (H^n(C^\bullet), 0)$, i.e., sending $C^\bullet$ to the cohomology sequence of $C^\bullet$ using zero boundary maps. Certainly, under this functor, quasi-isomorphisms because isomorphisms. Let's investigate further.

Since H sends quasi-isomorphisms to isomorphisms, we can factor it through $Q\colon \operatorname{Kom}\mathcal{C} \to D\mathcal{C}$; say $H = FQ$ with $F\colon D\mathcal{C} \to \operatorname{Kom}\mathcal{C}$. In fact, we can go further: complexes with zero differential form a complete category $\operatorname{Kom}_0\mathcal{C} \subseteq \operatorname{Kom}\mathcal{C}$, and H is a functor to $\operatorname{Kom}_0\mathcal{C}$, so really $F\colon D\mathcal{C} \to \operatorname{Kom}_0\mathcal{C}$. If F were an equivalence of categories, then this offers a simpler perspective to localization. The relevant condition here is *splitting* short exact sequences.

Definition 4.4 (Split SES)**.** A short exact sequence $0 \to A \to B \to C \to 0$ in an abelian category $\mathcal{C}$ is *split* if it is isomorphic to a sequence of the form $0 \to X \to X \oplus Y \to Y \to 0$, where the second and third maps are canonical. The category $\mathcal{C}$ is called *semisimple* if every short exact sequence splits.

Semisimplicity completely characterizes when the functor F discusses above is an equivalence.

Proposition 4.5. *If $\mathcal{C}$ is an abelian category, then $D\mathcal{C} \cong \operatorname{Kom}_0\mathcal{C}$ via the functor F if and only if $\mathcal{C}$ is semisimple.* ☐

Example 4.6. Every sequence in $\mathsf{Mod}_k = \mathsf{Vect}_k$ splits when k is a field, but the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \tag{4.3}$$

does not split since $\mathbb{Z}$ is torsion-free. This means that the derived category of vector spaces is equivalent to the category of cyclic complexes of vector spaces, but the same does not hold for abelian groups.

To see how this plays out, note that the sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0$ has cohomology $\mathbb{Z}/2$, as does the sequence $0 \to \mathbb{Z}/2 \to 0$, but the only map between these complexes is zero, so there is no quasi-isomorphism between them. Thus, they are distinct object in the derived category even though they map to the same object under the cohomology functor.

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